ECED 6400
Homework Assignment

1. Total internal reflection at isotropic-uniaxial medium interface

Consider an **extraordinary** plane electromagnetic wave, incident from a uniaxial crystal with dielectric constants $\epsilon_{1\perp}$ and $\epsilon_{1||}$ onto the interface $z = 0$, separating the crystal from a transparent isotropic medium with the dielectric constant ϵ_2. The optical axis of the crystal is **perpendicular to the interface**. Under what condition(s) among the parameters $\epsilon_{1\perp}$, $\epsilon_{1||}$, and ϵ_2, does total internal reflection take place? Repeat the problem for the optical axis of the crystal **parallel to the interface**.

Hint. You have to obtain conditions similar to $\epsilon_2 > \epsilon_1$ for isotropic media. Make sure that your result reduces to the conventional one in the isotropic medium limit.

2. Short pulse propagation in resonant linear absorbers: Energy losses

Starting, for example, from a general expression for a pulse envelope at any propagation distance within a homogeneously broadened linear absorber, address the following questions.

(a) Consider the pulse energy, defined as

$$W(z) \propto \int_{-\infty}^{\infty} dt |\mathcal{E}(t, z)|^2,$$

Using the properties of Fourier transforms, show that the energy attenuation factor, $\Gamma(z) = W(Z)/W(0)$, is given by

$$\Gamma(z) = \frac{\int_{-\infty}^{\infty} d\nu |\tilde{\mathcal{E}}(\nu)|^2 \, \exp\left(-\frac{2\alpha z}{1+\nu^2 T^2}\right)}{\int_{-\infty}^{\infty} d\nu |\tilde{\mathcal{E}}(\nu)|^2}.$$

Here $\tilde{\mathcal{E}}(\nu)$ is a spectral amplitude of the pulse at $z = 0$, and T is a dipole relaxation time.

(b) Specify to a Gaussian pulse, $\mathcal{E}(t, 0) \propto e^{-t^2/2T_p^2}$. Using the asymptotic method for integral evaluation outlined in the Appendix, show that for sufficiently long propagation distances, $\alpha z \gg 1$, the energy attenuation factor of an ultrashort Gaussian pulse ($T_p \ll T$) is

$$\Gamma_\infty(z) \simeq \exp\left(-\frac{2T_p}{T}\sqrt{2\alpha z}\right).$$

(c) Compare the behavior of $\Gamma_\infty(z)$ with that of a long pulse, $\Gamma_0(z) = e^{-\alpha z}$, in the long-term limit $\alpha z \gg 1$. How can you explain anomalously low energy loss rates of ultrashort pulses?

3. Symmetries of nonlinear optical susceptibilities in isotropic media

Use the invariance of $\chi^{(3)}_{ijkl}$ in isotropic media with respect to rotations by $45°$ around the z-axis to derive the following relation among the components of $\chi^{(3)}_{ijkl}$

$$\chi^{(3)}_{xxxx} = \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyyx} + \chi^{(3)}_{xyxy}.$$

4. Third harmonic generation: Beyond undepleted pump approximation

Consider the THG process for the case of plane wave geometry and perfect phase matching. In these conditions, the coupled wave equations derived in class simplify to

$$\frac{d\mathcal{E}_\omega}{dz} = \frac{3i\omega\chi^{(3)}_{eff}}{2n_\omega c}\,\mathcal{E}_{3\omega}\mathcal{E}^{*2}_\omega$$

and

$$\frac{d\mathcal{E}_{3\omega}}{dz} = \frac{3i\omega\chi^{(3)}_{eff}}{2n_{3\omega} c}\,\mathcal{E}^3_\omega$$

To simplify algebra slightly you may assume that $\chi^{(3)}_{eff}$ is real which works for lossless media.

(a) Introducing dimensionless amplitudes $\mathcal{A}_\omega$ and $\mathcal{A}_{3\omega}$ viz.,

$$\mathcal{E}_\omega = \sqrt{\frac{2I}{\epsilon_0 n_\omega c}}\mathcal{A}_\omega e^{i\phi_\omega}, \qquad \mathcal{E}_{3\omega} = \sqrt{\frac{2I}{\epsilon_0 n_{3\omega} c}}\mathcal{A}_{3\omega} e^{i\phi_{3\omega}},$$

derive the two integrals of motion,

$$\mathcal{A}^2_\omega + \mathcal{A}^2_{3\omega} = 1, \quad \text{(power conservation)},$$

and

$$\mathcal{A}_{3\omega}\mathcal{A}^3_\omega \cos\theta = \Gamma,$$

where $\theta = \phi_{3\omega} - 3\phi_\omega$.

(b) Consider the particular case $\theta = \pi/2$, implying that $\Gamma = 0$. Find and sketch the dependence of the fundamental and third harmonic modes on ζ. Assume that at $\zeta = 0$ all power resides with the fundamental.

(c) Estimate the efficiency of THG process in a 1 cm long glass sample, $n_{3\omega} \simeq n_\omega \sim 1.5$ by a cw laser with $P = 1$ W. Assume that $\lambda \sim 5 \times 10^{-5}$ cm and the laser light beam is tightly focused to a size of about 10^{-2} cm. How does the efficiency change if a pulsed laser source with $P = 1$ kW is used instead?

Appendix

1. Gaussian integrals

You may find useful the following Gaussian integrals

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{1}$$

and

$$\int_{-\infty}^{+\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right). \tag{2}$$

2. Laplace method for asymptotic evaluation of integrals

Consider an integral

$$I = \int_{-\infty}^{+\infty} dx e^{-\lambda f(x)},$$

for an arbitrary **real** function $f(x)$ in the limit of very large λ ($\lambda \to +\infty$). The main contribution to the integral comes from the neighborhood of the point at which f attains minimum. Let us call such a point x_0, and expand f in the vicinity of x_0 in a Taylor series up to the second order:

$$f(x) \simeq f(x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2.$$

Here at x_0

$$f'(x_0) = 0, \qquad f''(x_0) > 0.$$

The integral I is then

$$I \simeq e^{-\lambda f(x_0)} \int_{-\infty}^{+\infty} dx e^{-\lambda f''(x_0)(x-x_0)^2/2}.$$

Introducing the variable $s = x - x_0$, we can rewrite the integral as

$$I \simeq e^{-\lambda f(x_0)} \int_{-\infty}^{+\infty} ds e^{-\lambda f''(x_0)s^2/2}.$$

The integral on the r.h.s. can be evaluated using Eq. (1). The result is

$$I \simeq \sqrt{\frac{2\pi}{\lambda f''(x_0)}}\, e^{-\lambda f(x_0)}. \tag{3}$$

In case of multiple minima x_k, Eq. (3) is naturally generalized to

$$I \simeq \sum_k \sqrt{\frac{2\pi}{\lambda f''(x_k)}}\, e^{-\lambda f(x_k)}. \tag{4}$$