Pulse and beam propagation in linear media

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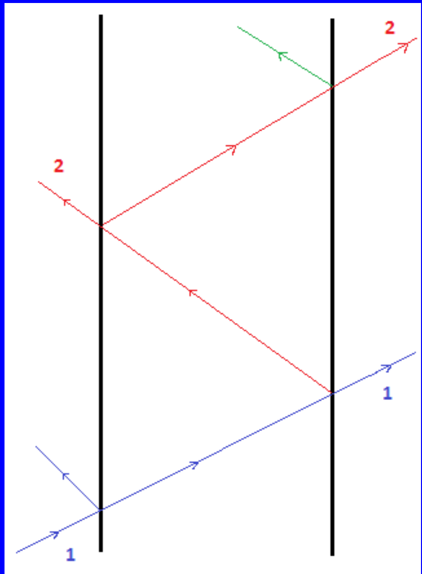
Outline

- Theory of optical dispersion and absorption
- Pulse propagation in linear media far from resonance
- Pulse propagation in linear media near resonance
- Homogeneous and inhomogeneous broadening
- Beer's absorption law & pulse area theorem
- Gaussian beam optics
- Angular spectrum representation of beams

Multi-wave interference phenomena. Fabry-Perot interferometer



TM-wave:

$$\mathbf{H}_s = H_s \mathbf{e}_y, \quad s = i, r, t.$$

Introduce reflection and transmission coefficients

$$r_{ij}; \quad t_{ij} \quad i,j = 1,2,3$$

Total reflected magnetic field:

$$\mathbf{H}_r = \mathbf{e}_y H_i \left(r_{12} + r_{23} t_{12} t_{21} e^{i2k_{2z}d} \sum_{s=0}^{\infty} r_{21}^s r_{23}^s e^{i2sk_{2z}d} \right).$$

Complex reflectivity, $\bar{r} \equiv E_r/E_i$:

$$\boxed{\bar{r} = \frac{r_{12} + r_{23} e^{2ik_{2z}d}}{1 + r_{12} r_{23} e^{2ik_{2z}d}}}$$

Reflectionless situation, $\bar{r} = 0$

$$r_{12} + r_{23} e^{2ik_{2z}d} = 0$$

Antireflection coatings

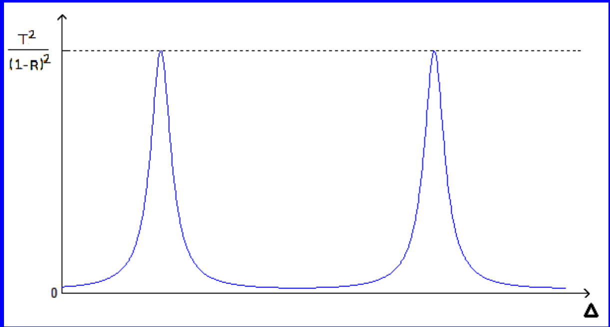
$k_{2z} = k_2 \Longrightarrow$ normal incidence

$$\boxed{d = \frac{\lambda}{4n_2}}$$

$$\boxed{n_2 = \sqrt{n_1 n_3}}$$

Applications: glare removal, night vision facilitation

Fabry-Perot interferometry



Transmission coefficient: $\mathscr{T} \equiv |E_t|^2/|E_i|^2$:

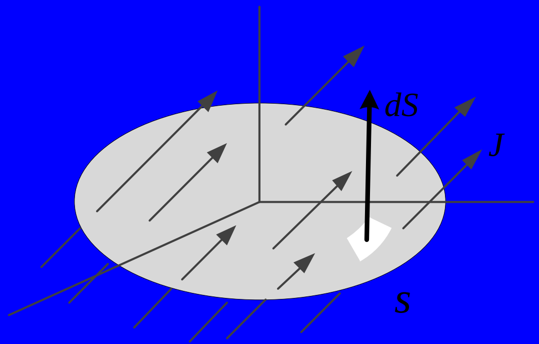
$$\boxed{\mathscr{T} = \frac{T^2}{(1-R)^2}\frac{1}{1+F\sin^2\Delta}}$$

$$F = \frac{4R}{(1-R)^2} \qquad R \approx 1 \Longrightarrow F \gg 1$$

Theory of optical dispersion and absorption

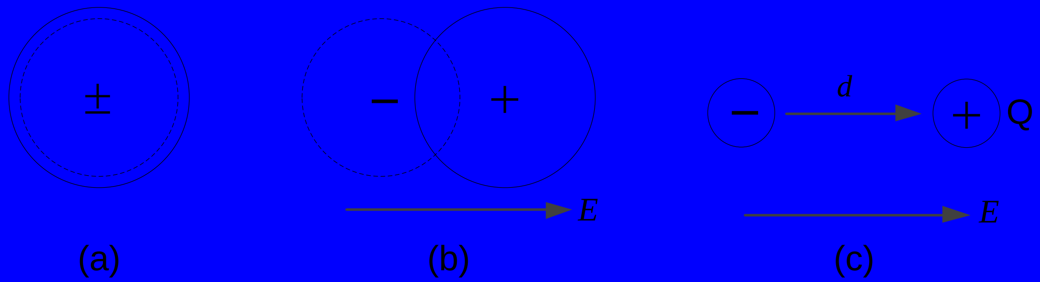
Ideal conductors: plenty of free charges

$$I = \int d\mathbf{S} \cdot \mathbf{J}, \qquad d\mathbf{S} = \mathbf{e}_n dS$$



$$\mathbf{J} = \sigma \mathbf{E}, \qquad \text{Ohm's law}$$

Ideal dielectrics: bound charges only:



$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P},$$

Linear, homogeneous isotropic dielectric with no spatial and temporal dispersion:

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}.$$

Lorentz oscillator model for real dielectrics

Model ingredients

- Response to a harmonic applied field $\mathbf{E}(t) = \mathbf{E}_\omega e^{-i\omega t}$;
- Bound electron $\Longleftrightarrow$ harmonic oscillator;
- f_{s} out of Z electrons per atom with ω_{s};
- $\mathbf{r}_{\mathrm{s}}$ displacement of the s-type electron;
- $\mathbf{F}_{\mathrm{r}} = -m\omega_{\mathrm{s}}^2\mathbf{r}_{\mathrm{s}}$, restoring force;
- $\mathbf{F}_{\mathrm{d}} = -2m\gamma_s\dot{\mathbf{r}}_{\mathrm{s}}$ radiation damping force.

Lorentz equation of motion

$$m\ddot{\mathbf{r}}_{\mathrm{s}} = -m\omega_{\mathrm{s}}^2\mathbf{r}_{\mathrm{s}} - 2m\gamma_{\mathrm{s}}\dot{\mathbf{r}}_{\mathrm{s}} - e\mathbf{E}_{\omega}e^{-i\omega t}$$

Driven solution:

$$\mathbf{r}_{\mathrm{s}}(t) = \mathbf{r}_{\mathrm{s}\omega}e^{-i\omega t}$$

Induced individual dipole moment:

$$\mathbf{p}_{\mathrm{s}} = -e\mathbf{r}_{\mathrm{s}}$$

The overall polarization:

$$\mathbf{P} = N\sum_{\mathrm{s}} f_s\mathbf{p}_{\mathrm{s}}$$

$$\mathbf{P} = \frac{Ne^2}{m} \sum_s \frac{f_s \mathbf{E}}{(\omega_s^2 - \omega^2 - 2i\omega\gamma_s)}$$

Comparing with

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi) \mathbf{E},$$

we arrive at the Lorentz dielectric permittivity:

$$\varepsilon(\omega) = 1 + \frac{Ne^2}{m} \sum_s \frac{f_s}{(\omega_s^2 - \omega^2 - 2i\omega\gamma_s)}$$

- Real part of $\varepsilon(\omega) \Longleftrightarrow$ dispersion;
- Imaginary part of $\varepsilon(\omega) \Longleftrightarrow$ absorption

Near-resonance behavior of optical susceptibility:

$$\omega \approx \omega_0 \neq 0,$$

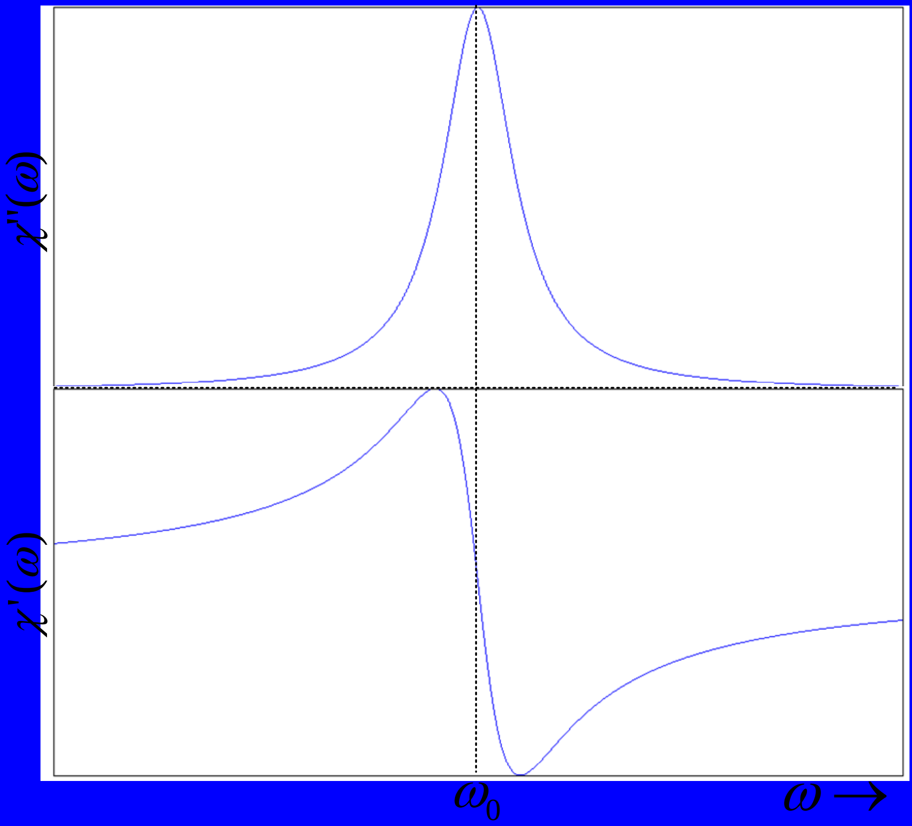
$$\varepsilon(\omega) = \underbrace{\varepsilon'(\omega)}_{\text{dispersion}} + i \underbrace{\varepsilon''(\omega)}_{\text{absorption}},$$

where

$$\varepsilon'(\omega) = \varepsilon_{\text{NR}}(\omega) + \frac{Ne^2 f_0}{2m\omega_0}\left[\frac{(\omega - \omega_0)}{(\omega - \omega_0)^2 + \gamma_0^2}\right],$$

and

$$\varepsilon''(\omega) = \frac{Ne^2 f_0}{2m\omega_0}\left[\frac{\gamma_0}{(\omega - \omega_0)^2 + \gamma_0^2}\right]$$


χ''(ω)
χ'(ω)
ω0
ω →

Real conductors;

Suppose f_0 electrons have $\omega_0 = 0$

$$\varepsilon_c(\omega) = \varepsilon_b(\omega) + i\frac{N f_0 e^2}{m\omega(2\gamma_0 - i\omega)}$$

$\varepsilon_b(\omega)$, bound electron contribution

$$\sigma(\omega) = \frac{N f_0 e^2}{m(2\gamma_0 - i\omega)}$$

Thus for realistic conducting materials,

$$\boxed{\varepsilon_c(\omega) = \varepsilon_b(\omega) + i(\sigma/\omega)}$$

High frequencies, $\omega \gg \max(\omega_s)$,

Plasma limit $\Longleftrightarrow$

$$\varepsilon_\mathrm{c}(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

with the plasma frequency

$$\omega_\mathrm{p} = \sqrt{\frac{NZe^2}{m}}$$

Drude model for noble metals; note that

$$\omega \geq \omega_p \Longrightarrow \varepsilon(\omega) \leq 0$$

Medium response to a monochromatic field:

$$\mathbf{P}(\mathbf{r},\omega) = \varepsilon_0 \chi(\omega) \mathbf{E}(\mathbf{r},\omega)$$

Pulse field

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathscr{E}(\mathbf{r},\omega) e^{-i\omega t}$$

$\mathscr{E}(\mathbf{r},\omega) \Longrightarrow$ the pulse spectral amplitude;

$$\mathscr{P}(\mathbf{r},\omega) = \varepsilon_0 \chi(\omega) \mathscr{E}(\mathbf{r},\omega)$$

Overall polarization field induced by the pulse:

$$\mathbf{P}(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{\infty} dt' \chi(t-t') \mathbf{E}(\mathbf{r},t')$$

Temporal nonlocality $\Longleftrightarrow$ frequency dispersion.

Pulse propagation in dispersive media: non-resonant case

Space-frequency representation

$$\tilde{\mathscr{D}}(\mathbf{r},\omega) = \varepsilon(\omega)\tilde{\mathscr{E}}(\mathbf{r},\omega),$$

Helmholtz equation

$$\nabla^2\tilde{\mathscr{E}} + \mu_0\varepsilon(\omega)\omega^2\tilde{\mathscr{E}} = 0$$

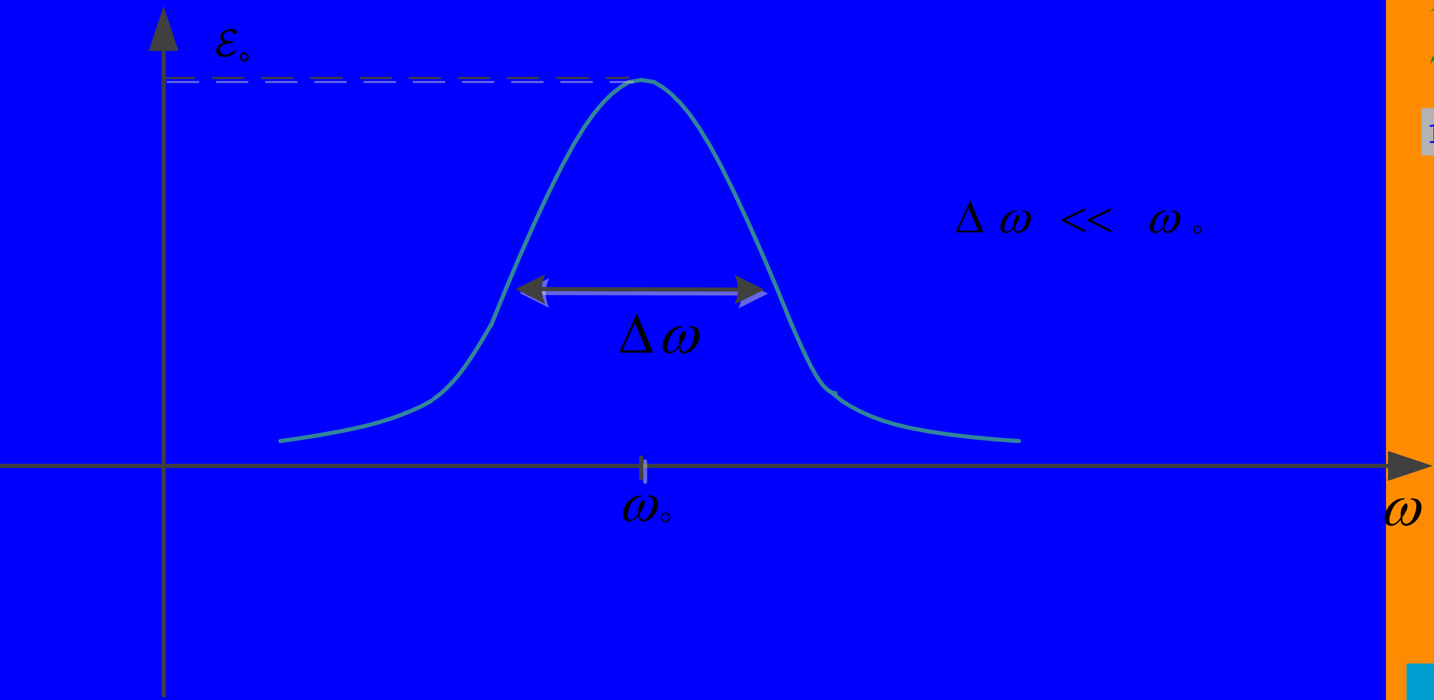
Pulse field

$$\tilde{\mathscr{E}}(\mathbf{r},\omega) = \mathbf{e}_x\tilde{\mathscr{E}}(\omega,z)e^{ik_0 z}$$

Slowly-varying envelope approximation (SVEA)

$$\partial_z\tilde{\mathscr{E}} \ll k_0\tilde{\mathscr{E}},$$

SVEA



$\Delta\omega << \omega_\circ$

SVEA evolution equation

$$2ik_0\partial_z\tilde{\mathscr{E}} + [k^2(\omega) - k_0^2]\tilde{\mathscr{E}} = 0,$$

where

$$k^2(\omega) = \mu_0\varepsilon(\omega)\omega^2.$$

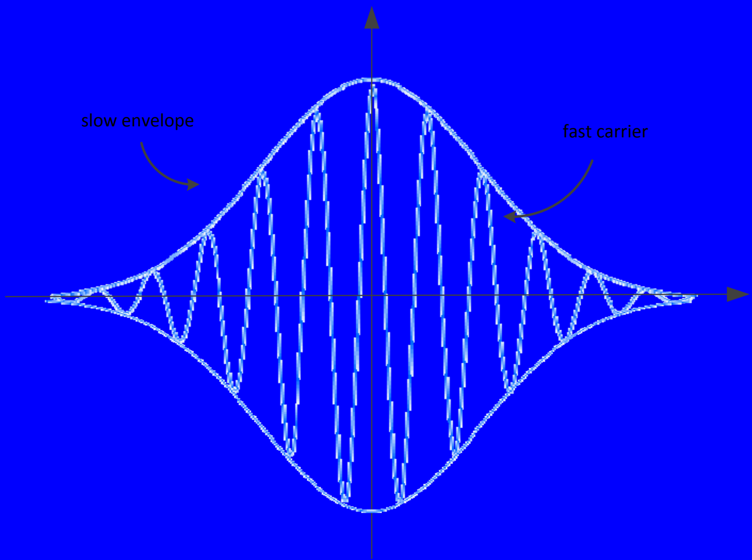
Narrow-band envelope approximation

$$\Delta\omega = 2|\omega_{\max} - \omega_0| \ll \omega_0,$$

$\omega_{max} \Longleftrightarrow$ highest frequency within the envelope.

$$k(\omega) \simeq k_0 + \underbrace{k'(\omega_0)}_{k_1}(\omega - \omega_0) + \frac{1}{2!}\underbrace{k''(\omega_0)}_{k_2}(\omega - \omega_0)^2$$

Fast carrier+slow envelope picture:



$$\mathbf{E}(t,z) = \mathbf{e}_x \underbrace{e^{i(k_0 z - \omega_0 t)}}_{carrier\ wave} \underbrace{\int_{-\infty}^{+\infty} d\omega' e^{-i\omega' t} \tilde{\mathscr{E}}(\omega', z)}_{slow\ envelope} .$$

Space-time representation

$$2i(\partial_z \mathscr{E} + k_1 \partial_t \mathscr{E}) - k_2 \partial_{tt}^2 \mathscr{E} = 0.$$

in the reference frame moving with the pulse;

$$\zeta = z; \quad \tau = t - k_1 z,$$

Paraxial wave equation in time:

$$\boxed{2i\partial_\zeta \mathscr{E} - k_2 \partial_{\tau\tau}^2 \mathscr{E} = 0}$$

Space-time analogy

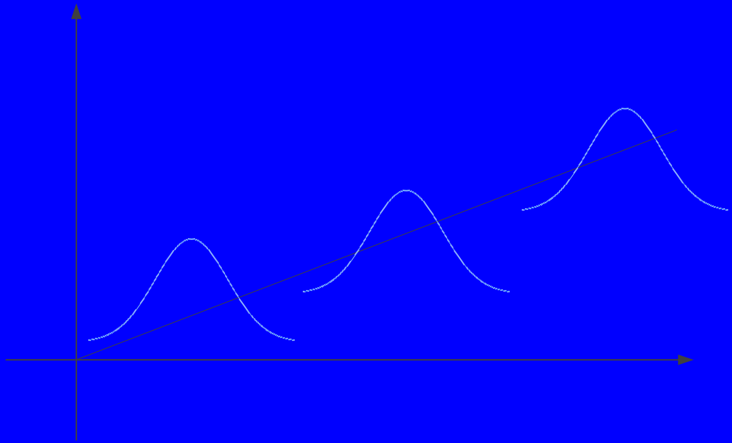
- $\tau \Longleftrightarrow x$, $T_p \Longleftrightarrow w_0$;
- $L_{\mathrm{dis}} = T_p^2/k_2 \Longleftrightarrow L_{\mathrm{D}} = k w_0^2$.

in dispersionless media $k_2 = 0$,

$$\partial_\zeta \mathscr{E} = 0,$$

implying shape-invariant pulse propagation:

$$\mathscr{E}(t,z) = \mathscr{E}_0(t - z/v_g), \qquad v_g \equiv k_1^{-1}$$



Evolution at length scales $\zeta \ll L_{\text{dis}}$

Pulse propagation in resonant linear absorbers

Lorentz oscillator model for the medium:

$$\partial_t^2 x + 2\gamma\partial_t x + \omega_0^2 x = -eE/m,$$

Electric and dipole fields:

$$E(z,t) = \frac{1}{2}[\mathscr{E}(z,t)e^{i(kz-\omega t)} + c.c]$$

and

$$ex(z,t) = \frac{1}{2}[d_0\sigma(z,t)e^{i(kz-\omega t)} + c.c]$$

SVEA for the field and dipole envelopes:

$$\partial_z \mathscr{E} \ll k\mathscr{E}, \quad \partial_t \mathscr{E} \ll \omega \mathscr{E}$$

$$\partial_t \boldsymbol{\sigma} \ll \boldsymbol{\omega\sigma}.$$

SVEA evolution of the dipole envelope:

$$\partial_t \boldsymbol{\sigma} = -(\gamma + i\Delta)\boldsymbol{\sigma} + i\Omega,$$

$$\Omega = -e\mathscr{E}/2m\omega x_0 \Longleftrightarrow \mathscr{E} \text{ in frequency units.}$$

Homogeneous line broadening

Switched off cw field,

$$\Omega(t) = \theta(-t)\Omega_0(z)$$

Atomic dipole relaxation:

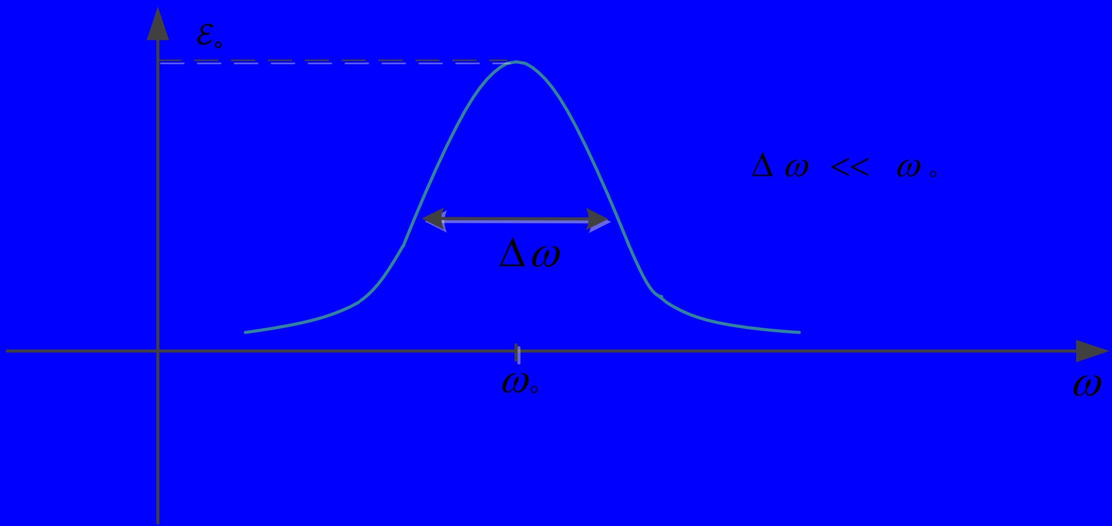
$$\sigma(t,z) = \sigma(0,z)\theta(t)e^{-\gamma t}e^{i\omega_0 t}$$

$$\tilde{\sigma}(\omega,z) \overset{FT}{\Longleftrightarrow} \sigma(t,z)$$

Absorption spectrum

$$S_0(\omega,z) \propto |\tilde{\sigma}(\omega,z)|^2$$

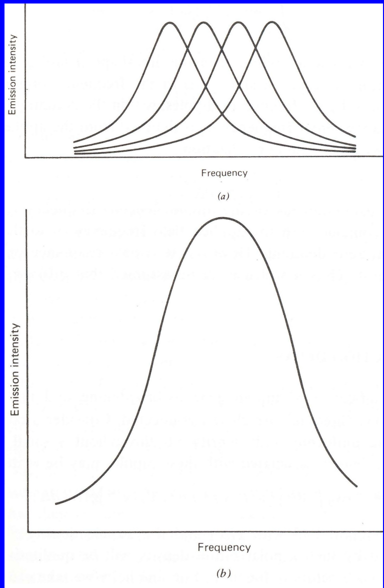
$$S_0(\omega, z) \propto \frac{|\sigma(0,z)|^2}{(\omega-\omega_0)^2+\gamma^2}$$



$\Delta\omega \equiv \gamma = 1/T \iff$ homogeneous line width

Inhomogeneous line broadening

Distribution of resonance frequencies $\Longrightarrow$ atom emisssion/absorption spectrum broadening



$$P(t,z) = \frac{1}{2}[\mathscr{P}(t,z)e^{i(kz-\omega t)} + c.c],$$

where

$$\mathscr{P}(t,z) = Nd_0\langle \boldsymbol{\sigma}(t,z,\omega_0)\rangle,$$

the averaging defined as

$$\langle \boldsymbol{\sigma}(t,z,\omega_0)\rangle = \int_0^\infty d\omega_0\, g(\omega_0)\boldsymbol{\sigma}(t,z,\omega_0)$$

normalization

$$\int_0^\infty d\omega_0\, g(\omega_0) = 1$$

Sharply peaked distribution around $\overline{\omega}_0$:

$$g(\omega_0) \simeq g(\omega_0 - \overline{\omega}_0) = g(\Delta)$$

Averaging over frequency detuning Δ:

$$\begin{aligned}\int_0^\infty d\omega_0 g(\omega_0)(\ldots) &= \int_{-\overline{\omega}_0}^\infty d\Delta\, g(\Delta)(\ldots) \\ &\simeq \int_{-\infty}^\infty d\Delta\, g(\Delta)(\ldots) \quad (1)\end{aligned}$$

Thus with inhomogeneous broadening

$$\mathscr{P}(t,z) = N d_0 \int_{-\infty}^\infty d\Delta\, g(\Delta)\, \sigma(t,z,\Delta)$$

Maxwell-Lorentz pulse evolution equations

Wave equation for EM field:

$$\partial_{zz}^2 E - c^{-2}\partial_{tt}^2 E = \mu_0 \partial_{tt}^2 P$$

Medium polarization:

$$P = -Ne\langle x\rangle$$

SVEA:

$$\partial_z \mathscr{E} \ll k\mathscr{E}, \qquad \partial_t \mathscr{E} \ll \omega\mathscr{E}$$

and

$$\partial_t \sigma \ll \omega\sigma$$

EM field envelope evolution (reduced Maxwell):

$$\partial_\zeta \Omega = i\kappa\langle\sigma\rangle,$$

Dipole moment envelope evolution (Lorentz):

$$\partial_\tau \sigma = -(\gamma + i\Delta)\sigma + i\Omega.$$

in the moving reference frame:

$$\zeta = z \quad \tau = t - z/c$$

Classical area theorem

Solution to MBE

$$\mathscr{E}(t,z) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \mathscr{E}(t',0) \int_{-\infty}^{\infty} d\omega\, e^{i\omega(t'-t)} \times \exp[i\omega z/c - \kappa \mathscr{R}(\omega) z] \quad (2)$$

where

$$\mathscr{R}(\omega) = \left\langle \frac{1}{\gamma + i(\Delta - \omega)} \right\rangle$$

Introducing the area:

$$A(z) = \int_{-\infty}^{\infty} dt\, \mathscr{E}(t,z)$$

Area theorem:

$$A(z) = A_0 e^{-\alpha z/2} e^{i\beta z/2}$$

where

$$\alpha = \left\langle \frac{2\kappa\gamma}{\gamma^2 + \Delta^2} \right\rangle,$$

and

$$\beta = \left\langle \frac{2\kappa\Delta}{\gamma^2 + \Delta^2} \right\rangle.$$

Regardless of pulse profile $\Longrightarrow$ exponential decay of the pulse area $\Longrightarrow$ universal global dynamics!

Beer's absorption law

Long pulses

$$T_p \gg \max(T_0, T_\Delta)$$

adiabatically eliminating atomic variables:

$$\partial_\zeta \mathscr{E} = -\kappa \left\langle \frac{1}{\gamma + i\Delta} \right\rangle \mathscr{E}$$

$$\mathscr{E}(t, z) = e^{-\alpha z/2} e^{i\beta z/2} \mathscr{E}_0(t - z/c)$$

$$L_B = \alpha^{-1}, \Longleftrightarrow \text{Beer's length}$$

Paraxial optics: Gaussian beams

Monochromatic field evolution in free space:

$$\mathbf{E}(\mathbf{r},t) = \mathscr{E}(\mathbf{r},\omega)e^{-i\omega t},$$
$$\mathbf{H}(\mathbf{r},t) = \mathscr{H}(\mathbf{r},\omega)e^{-i\omega t}$$

Maxwell's equations:

$$\nabla \times \mathscr{E} = i\mu_0\omega\mathscr{H},$$
$$\nabla \times \mathscr{H} = -i\varepsilon_0\omega\mathscr{E},$$

and

$$\nabla \cdot \mathscr{E} = 0,$$
$$\nabla \cdot \mathscr{H} = 0.$$

Back
Close

Deriving Helmholtz equation for electric field

$$\nabla^2 \mathscr{E} + k^2 \mathscr{E} = 0,$$

where $k = \omega / c$.

Seeking a plane polarized beam-like solution:

$$\mathscr{E} = \mathbf{e}_y \mathscr{E}(x, z) e^{ikz}.$$

Paraxial approximation

$$\partial_z \mathscr{E} \ll k \mathscr{E},$$

$$\boxed{2ik\partial_z \mathscr{E} + \partial^2_{xx} \mathscr{E} = 0}$$

Gaussian beam solution

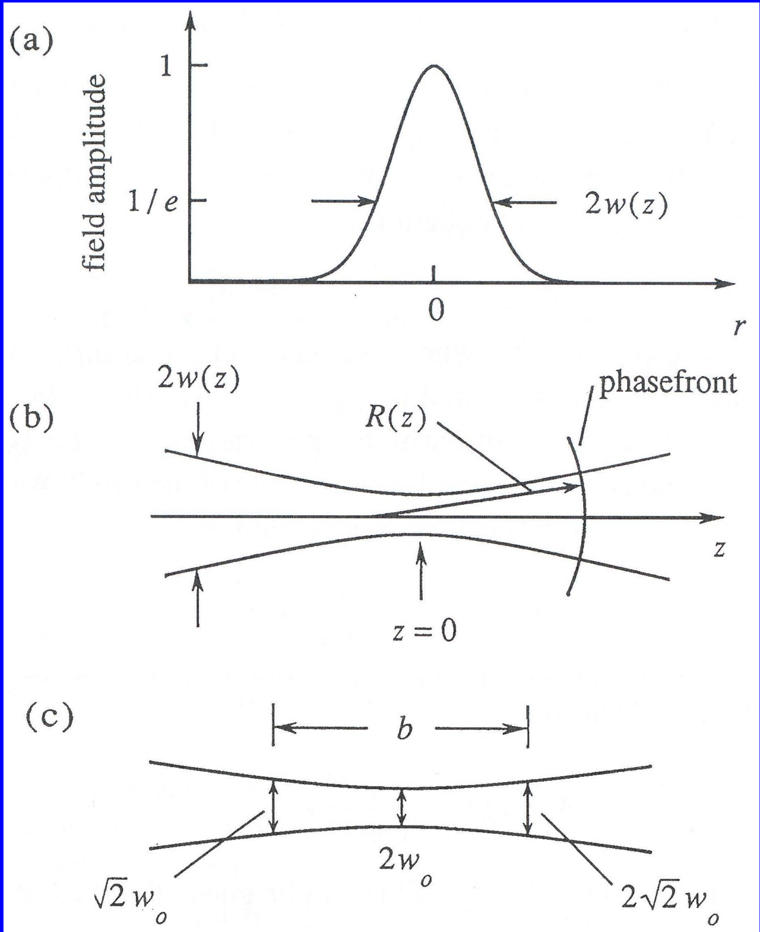
$$\mathscr{E}(x,z) = \frac{\mathscr{E}_0}{\sqrt{1+i\zeta}} \exp\left[-\frac{x^2}{2w_0^2(1+i\zeta)}\right],$$

in a complex form where

$$\zeta = z/z_R, \qquad z_R = kw_0^2.$$

or in a real form

$$\mathscr{E}(x,z) = \mathscr{E}_0\sqrt{\frac{w_0}{w(z)}}e^{i\Phi(z)} \exp\left[\frac{ikx^2}{2R(z)}\right] \times \exp\left[-\frac{x^2}{2w^2(z)}\right]. \tag{3}$$


(a)
field amplitude
1
1/e
2w(z)
0
r
2w(z)
(b)
R(z)
phasefront
z
z = 0
(c)
b
2w
o
√2 w
o
2√2 w
o

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Parameters:

- beam width $w(z) = w_0\sqrt{1 + z^2/z_R^2}$;
- wavefront curvature radius $R(z) = z(1 + z_R^2/z^2)$;
- accrued phase, $\Phi(z) = -\frac{1}{2}\arctan(z/z_R)$.
- wavefront phase, $\Psi(x,z) = \Phi(z) + \frac{kx^2}{2R(z)}$

Wavefront is defined as:

$$\Psi(x,z) = const.$$

Angular spectrum representation of the beam

Introduce the angular spectrum decomposition:

$$\mathscr{E} = \mathbf{e}_y \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_x dk_z \tilde{\mathscr{A}}(k_x, k_z) e^{i(k_x x + k_z z)}.$$

Helmholtz equation yields

$$\tilde{\mathscr{A}}(k_x, k_z)(-k_x^2 - k_z^2 + k^2) = 0.$$

implying that

$$\tilde{\mathscr{A}}(k_x, k_z) = \mathscr{A}(k_x)\delta(k_x^2 + k_z^2 - k^2).$$

circle of radius k in the $k-$space:

$$k_x^2 + k_z^2 = k^2 \implies k_z = \sqrt{k^2 - k_x^2}$$

Thus

$$k_z = \begin{cases} \sqrt{k^2 - k_x^2}, & k_x < k \\ \pm i\sqrt{k_x^2 - k^2}, & k_x > k \end{cases}$$

Angular spectrum in the half-space $z \geq 0$

$$\mathscr{E} = \underbrace{\mathbf{e}_y \int_{k_x<k} dk_x \mathscr{A}(k_x) e^{i(k_x x + \sqrt{k^2 - k_x^2}\, z)}}_{\textit{homogeneous waves}} + \underbrace{\mathbf{e}_y \int_{k_x>k} dk_x \mathscr{A}(k_x) e^{ikx} e^{-\sqrt{k_x^2 - k^2}\, z}}_{\textit{evanescent waves}}. \quad (4)$$

Homogeneous waves only

$$\mathscr{E} = \mathbf{e}_y \int_{k_x<k} dk_x\, \mathscr{A}(k_x) e^{i(k_x x + \sqrt{k^2-k_x^2}\, z)}$$

Paraxial approximation (beams):

$$\sqrt{k^2 - k_x^2} \simeq k - \frac{k_x^2}{2k},$$

The beam field:

$$\mathscr{E} = \mathbf{e}_y e^{ikz} \int_{-\infty}^{+\infty} dk_x\, \mathscr{A}(k_x)\, e^{ik_x x} \exp\left(-\frac{ik_x^2 z}{2k}\right).$$